extreme sides turn about fixed points, the common vertex describes a curve of the n^{th} order, whose equation is

$$axB_1c_1D_1xB_2c_2D_2xB_3 \; . \; . \; . \; ex = 0.$$

Grassmann has also successfully applied his methods to the theory of projective ranges and pencils, and to the theory of harmonic poles and polars. In both cases was he led to noteworthy generalizations.

It is, however, in the science of rational mechanics that the application of Grassmann's theories and methods, in combination with those of Sir Wm. R. Hamilton, will probably prove of the greatest importance. As a valuable step in this direction, the reader might well be referred to the *Vector Analysis* of Professor J. Willard Gibbs (New Haven, 1881–1884), were it not for the fact that the words "not published" which appear on the title-page would seem to exclude that work from general circulation.

THE ATTRACTION OF A RIGHT CIRCULAR CYLINDER ON A PARTICLE.

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I propose to give here a more elaborate treatment of this problem, which is one of the illustrations in my paper, read before the Mathematical Section of the Philosophical Society of Washington entitled: *Can the Attraction of a Finite Mass be Infinite?*

Let $m =$ the mass of attracted particle,

$\vartheta =$ density of attracting cylinder,

$r =$ radius of attracting cylinder,

$h =$ height of attracting cylinder,

$d =$ distance of attracted particle from axis of cylinder,

$e =$ elevation of attracted particle above base of cylinder.

Assume the vertical line through the attracted particle for the axis of cylindrical co-ordinates in which $\rho =$ horizontal radius vector, $v =$ horizontal angle with central radius vector, $z =$ elevation above (or depression below) attracted particle; then, since

$$\sqrt{(\rho^2 + z^2)} = \text{distance of any mass element from particle}$$

and $\quad \vartheta\rho d\rho dv dz =$ mass element,

we have $\quad m\vartheta\dfrac{\rho d\rho dv dz}{\rho^2 + z^2} =$ attraction of a mass element on the particle

and $m\vartheta\dfrac{\rho d\rho dv dz}{\rho^2 + z^2}\cdot\dfrac{\rho\cos v}{\sqrt{(\rho^2+z^2)}} =$ component of this along central radius vector.

The sum of these components is the whole attraction; hence

$$A = m\vartheta\int_{d-r}^{d+r}\rho^2 d\rho\int_{-v}^{v}dv\cos v\int_{-e}^{h-e}\frac{dz}{\sqrt{(\rho^2+z^2)^3}}, \text{ where } \cos v = \frac{\rho^2+d^2-r^2}{2\rho d},$$

$$= 2\,m\vartheta\int_{d-r}^{d+r}\rho^2 d\rho\,.\sin v\,.\frac{1}{\rho^2}\left(\frac{h-e}{\sqrt{[\rho^2+(h-e)^2]}}+\frac{e}{\sqrt{(\rho^2+e^2)}}\right), \text{ where}$$

$$\sin v = \frac{1}{2\rho d}\sqrt{[(d+r)^2-\rho^2][\rho^2-(d-r)^2]},$$

$$= m\frac{\vartheta}{d}\int_{d-r}^{d+r}\frac{d\rho}{\rho}\sqrt{[(d+r)^2-\rho^2][\rho^2-(d-r)^2]}\left(\frac{h-e}{\sqrt{[\rho^2+(h-e)^2]}}+\frac{e}{\sqrt{(\rho^2+e^2)}}\right). \tag{1}$$

The first term evidently expresses the attraction of the portion of the cylinder above the particle, which we shall denote A_{h-e}, and the second term that of the portion below the particle $= A_e$, and by simply replacing $h-e$ by e we obtain the second from the first and have $A = A_{h-e} + A_e$. To simplify, place in A_e

$$\begin{aligned} a_e &= \sqrt{[(d+r)^2+e^2]},\\ b_e &= \sqrt{[(d-r)^2+e^2]},\\ c_e &= \sqrt{(a_e^2-b_e^2)} = 2\sqrt{(rd)}; \end{aligned} \tag{2}$$

hence in A_{h-e}

$$\begin{aligned} a_{h-e} &= \sqrt{[(d+r)^2+(h-e)^2]},\\ b_{h-e} &= \sqrt{[(d-r)^2+(h-e)^2]},\\ c_{h-e} &= c_e = 2\sqrt{(rd)}; \end{aligned}$$

then

$$A_e = m\frac{\vartheta e}{d}\int_{d+r}^{d-r}\frac{d\rho}{\rho}\sqrt{\frac{[a_e^2-(\rho^2+e^2)][(\rho^2+e^2)-b_e^2]}{\rho^2+e^2}}. \tag{3}$$

Let

$$\tan\varphi = \frac{a_e}{b_e}\sqrt{\frac{\rho^2+e^2-b_e^2}{a_e^2-\rho^2-e^2}}$$

and the new limits are $\lrcorner$ (quadrant) and o, and putting

$$\Delta\varphi = \sqrt{\left(1 - \frac{c_e^2}{a_e^2}\sin^2\varphi\right)}$$

$$= \sqrt{(a_e^2\cos^2\varphi + b_e^2\sin^2\varphi)}\ \left(= \text{the elliptic } \Delta \text{ function to the modulus } \frac{c_e}{a_e}\right),$$

we have also $$\sqrt{(\rho^2 + e^2)} = \frac{b_e}{\Delta\varphi},$$

$$\sqrt{[a_e^2 - (\rho^2 + e^2)]\,[(\rho^2 + e^2) - b_e^2]} = \frac{a_e b_e}{\Delta\varphi^2}\sqrt{\left(\Delta\varphi^2 - \frac{b_e^2}{a_e^2}\right)(1 - \Delta\varphi^2)},$$

$$\rho d\rho = \frac{b_e^2 c_e^2 \sin\varphi\cos\varphi d\varphi}{a_e^2\Delta\varphi^4} = \frac{b_e^2 c_e^2 d\varphi}{a_e^2\Delta\varphi^4}\cdot\frac{a_e^2}{c_e^2}\sqrt{(1 - \Delta\varphi^2)\left(\Delta\varphi^2 - \frac{b_e^2}{a_e^2}\right)},$$

$$\frac{d\rho}{\rho} = \frac{b_e^2 d\varphi\sqrt{(1 - \Delta\varphi^2)\left(\Delta\varphi^2 - \frac{b_e^2}{a_e^2}\right)}}{\left(\frac{b_e^2}{\Delta\varphi^2} - e^2\right)\Delta\varphi^4};$$

hence

$$A_e = m\frac{\vartheta e}{d}\int_0^{\lrcorner}\frac{d\rho\sqrt{(1 - \Delta\varphi^2)\left(\Delta\varphi^2 - \frac{b_e^2}{a_e^2}\right)}}{\left(1 - \frac{e^2}{b_e^2}\Delta\varphi^2\right)\Delta\varphi^2}\cdot\frac{\Delta\varphi}{b_e}\cdot\frac{a_e b_e}{\Delta\varphi^2}\sqrt{(1 - \Delta\varphi^2)\left(\Delta\varphi^2 - \frac{b_e^2}{a_e^2}\right)}$$

$$= m\vartheta a_e\frac{e}{d}\int_0^{\lrcorner} d\rho\frac{(1 - \Delta\varphi^2)\left(\Delta\varphi^2 - \frac{b_e^2}{a_e^2}\right)}{\left(1 - \frac{e^2}{b_e^2}\Delta\varphi^2\right)\Delta\varphi^3}$$

$$= -m\vartheta\frac{b_e^2 e}{a_e d}\int_0^{\lrcorner}\frac{d\rho}{\Delta\varphi^3} + m\vartheta\frac{b_e^2 a_e}{ed}\int_0^{\lrcorner}\frac{d\varphi}{\Delta\varphi} - m\vartheta\frac{(d^2 - r^2)^2}{a_e ed}\int_0^{\lrcorner}\frac{d\rho}{\left(1 - \frac{e^2}{b_e^2}\Delta\varphi^2\right)\Delta\varphi}$$

or, since $$\int_0^{\phi}\frac{d\varphi}{\Delta\varphi^3} = \frac{a_e^2}{b_e^2}\left(\int_0^{\phi} d\varphi\Delta\varphi - \frac{c_e^2\sin\varphi\cos\varphi}{a_e^2\Delta\varphi}\right)$$

$$A_e = m\vartheta\frac{b_e^2 a_e}{ed}\left(\int_0^{\lrcorner}\frac{d\varphi}{\Delta\varphi} - \frac{e^2}{b_e^2}\int_0^{\lrcorner} d\varphi\Delta\varphi - \frac{(d + r)^2}{a_e^2}\int_0^{\lrcorner}\frac{d\varphi}{\left[1 + \frac{c_e^2}{a_e^2}\ \frac{e}{(d - r)^2}\sin^2\varphi\right]\Delta\varphi}\right)$$

$$= \vartheta m \frac{b_e^2 a_e}{ed}\left[F - \frac{e^2}{b_e^2} E - \frac{(d+r)^2}{a_e^2}\, \Pi\left(\frac{c_e^2}{a_e^2} \cdot \frac{e^2}{(d-r)^2}\right)\right], \tag{4}$$

where, by Legendre's notation, F, E, and Π are quadrantal elliptic integrals of the first, second, and third species respectively. The modulus is $\frac{c_e}{a_e}$ and the parameter $\frac{c_e^2}{a_e^2} \cdot \frac{e^2}{(d-r)^2}$.

Because the integral of the third species is quadrantal it may be expressed (as was first shown by Legendre) in terms of integrals of the first and second species, and I might apply his formula of reduction at once. I prefer, however, to make the reduction by means of Jacobi's theorem for exchanging amplitude and parameter (in Jacobi's sense), making use also of a new system of notation which I have used with advantage for many years.* In this notation I denote the modulus by γ and the complementary modulus by β (instead of k, k' by Jacobi). Taking the integral of the first species,

$$u = \int_0^\phi \frac{d\varphi}{\sqrt{(1-\gamma^2 \sin^2\varphi)}} = \int_0^\phi \frac{d\varphi}{\Delta\varphi},$$

I write

$$u = \varphi_\gamma. \tag{5}$$

If $\gamma = 0$, we have $u = \varphi_0 = \varphi$, and the integral is identical with its amplitude, and for any value of the modulus it may be looked upon (as the notation implies) as a modified amplitude.

Of (5) I take the inverse thus :—

$$u_{-\gamma} = \varphi. \tag{6}$$

It should be noted here that u represents an integral and $u_{-\gamma}$ an angle of which trigonometric functions may be taken. Thus we have $\sin u_{-\gamma}$, $\cos u_{-\gamma}$, $\tan u_{-\gamma}$, and in addition the Δ function $\Delta u_{-\gamma} = \sqrt{(1 - \gamma^2 \sin^2 u_{-\gamma})}$, which are Jacobi's elliptic functions, and which he denoted as sin am u, cos am u, tan am u, Δ am u, a notation much complained of for its clumsiness. Gudermann proposed the notation sn u, cn u, tn u, dn u. This is short and convenient to write, but is apt to create the false impression that these quantities are no longer *ordinary* trigonometric functions. Besides, neither Jacobi's nor Gudermann's notation can make the distinction between different moduli conveniently.

*I have explained and used this notation in a paper on the quadric transformation read before the Mathematical Section of the Philosophical Society of Washington.

To express the quadrantal integral we might write

$$\left(\frac{\pi}{2}\right)_\gamma = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\Delta\varphi}.$$

But it is evidently desirable for this purpose to have a single symbol for $\frac{\pi}{2}$ and I have adopted the symbol $\lrcorner$ (quadrant) in the place of $\frac{\pi}{2}$, a notation which is certainly typical of its meaning.

Thus,

$$\lrcorner_\gamma = \int_0^{\lrcorner} \frac{d\varphi}{\Delta\varphi} \quad (= K \text{ of Jacobi}),$$

and the integral to the complementary modulus is

$$\lrcorner_\beta = \int_0^{\lrcorner} \frac{d\varphi}{\sqrt{(1 - \beta^2 \sin^2 \varphi)}} = \int_0^{\lrcorner} \frac{d\varphi}{\Delta(\varphi_\beta)_{-\beta}} \quad (= K' \text{ of Jacobi}).$$

The mode of indicating the modulus with the Δ function, employed here, should in strictness be used altogether. However, it is sufficient to indicate it only for a new modulus.

It is important to notice that if n is an integer, then, and only then,

$$(n \lrcorner)_\gamma = n \lrcorner_\gamma;$$

thus, the integral

$$(\tfrac{1}{2} \lrcorner)_\gamma = \int_0^{\frac{1}{2}\lrcorner} \frac{d\varphi}{\Delta\varphi}$$

should not be confounded with $\tfrac{1}{2} \lrcorner_\gamma = \tfrac{1}{2} \int_0^{\lrcorner} \frac{d\varphi}{\Delta\varphi}.$

[TO BE CONTINUED.]